Arithmetic derivative by Lucy

Introduction and Definitions

The arithmetic derivative is generally denoted as a function over the integers but can easily be extended to other numbers) to begin, and to get into the spirit we will start by proving some basic facts about the arithmetic derivative that are currently known. Even though we will later mostly be disregarding this definition in order to generalise the function.

First the formal definition of the function, denoted as x′ for now

$$p' = 1 \leftarrow p \in \mathbb{P}$$

In plain English, if p is a prime number, the arithmetic derivative of p is one.

$$(ab)' = a(b') + (a')b \leftarrow a, b \in \mathbb{Z}$$

Again, in plain English, the arithmetic derivative of a times b, is the arithmetic derivative of a, times b, add the arithmetic derivative of b, times a. i.e. a direct analogy to the Leibniz rule.

We will first have a proof that the arithmetic derivative does indeed act like a function over the positive integers (there is a unique arithmetic derivative for each positive integer)

Lemma: The arithmetic derivative acts as a function over the integers

Proof:

Suppose integer A has the unique prime factorisation $p_1 * p_2 * p_3 * \dots p_k$. One can attest that A′ = $p'_1 * p_2 * p_3 * \dots p_k + p_1 * p'_2 * p_3 * \dots p_k + p_1 * p'_2 * p'_3 * \dots p_k$+$p_1 * p_2 * p_3 * \dots p'_k$. (via direct application of the Leibnitz rule on multiple factors – to see this another way, imagine each factor is the function x, and apply the Leibniz rule for functions). To continue, the arithmetic derivative of each of these prime factors is one, per the definition of the arithmetic derivative. One can now see that the arithmetic derivative of A can be defined in terms of A.

$$A' = p_2 * p_3 * \dots p_k + p_1 * p_3 * \dots p_k + p_1 * p_2 * \dots p_k + p_1 * p_2 * p_3 * \dots p_{k-1}$$

$$A' = p_1 * p_2 * p_3 * \dots p_k * \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \dots + \frac{1}{p_k}\right) = A * \left(\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \dots + \frac{1}{p_k}\right)$$

Expanding the brackets and remembering that $p' = 1 \leftarrow p \in \mathbb{P}$ we can immediately see this is true by expanding the brackets.

Now suppose that A has any factors, prime or composite $f_1 * f_2 \dots f_k = A$. Using the definition derived above:

$$A' = (f_1 * f_2 \dots f_k)' = (f'_1 f_2 \dots f_k + f_1 f'_2 \dots f_k + f'_2 f_1 \dots f_k)$$

We can then use our summation form of the arithmetic derivative, defined above, on each f with p_{nfm} being the nth prime factor of f_m.

$$\begin{aligned} A' = (f_1 * f_2 \dots f_k)' \\ &= (f_1 f_2 \dots f_k * \left(\frac{1}{p_{1f1}} + \frac{1}{p_{2f1}} + \frac{1}{p_{3f1}} \dots + \frac{1}{p_{kf1}}\right) \\ &+ f_1 f_2 \dots f_k \left(\frac{1}{p_{1f2}} + \frac{1}{p_{2f2}} + \frac{1}{p_{3f2}} \dots + \frac{1}{p_{kf2}}\right) + \cdots \\ &+ f_1 f_2 \dots f_k \left(\frac{1}{p_{kf3}} + \frac{1}{p_{2f}3} + \frac{1}{p_{3f3}} \dots + \frac{1}{p_{kf3}}\right) \end{aligned}$$

All of the components above share the common factor of $f_1 f_2 \dots f_k (= A)$. We can now see our equality is equal to

$$A' = A * \left(\frac{1}{p_{1f1}} + \frac{1}{p_{2f1}} + \frac{1}{p_{3f1}} \dots + \frac{1}{p_{kf1}}\right) + A * \left(\frac{1}{p_{1f2}} + \frac{1}{p_{2f2}} + \frac{1}{p_{3f2}} \dots + \frac{1}{p_{kf2}}\right)$$

$$\dots + A * \left(\frac{1}{p_{1fk}} + \frac{1}{p_{2fk}} + \frac{1}{p_{3fk}} \dots + \frac{1}{p_{4fk}}\right)$$

$$A' = \sum_{n=1}^{k} A * (\frac{1}{p_{1f1}} + \frac{1}{p_{2f1}} + \frac{1}{p_{3f1}} \dots + \frac{1}{p_{kf2}} + \frac{1}{p_{1f2}} + \frac{1}{p_{2f2}} + \frac{1}{p_{3f2}} \dots + \frac{1}{p_{kf2}} \dots + \frac{1}{p_{1fk}} + \frac{1}{p_{2fk}} + \frac{1}{p_{3fk}} \dots + \frac{1}{p_{kfk}})$$

Remembering the fact that all these f's are factors of A, their prime factors summed will be the same as A's summed, As a example to ease the mind. If $4 * 6 = 24$ then $(sum\ of\ prime\ factors\ of\ 4 * 6) = (sum\ of\ prime\ factors\ of\ 24) = 9$. The same argument is made with their reciprocals.

As this argument can be made with any arbitrary factorisation of A. There must be a single unique arithmetic derivative of A over the integers and thus the arithmetic derivative behaves like a function over them. This is a simple argument but it can be show the arithmetic derivative acts as a function over a UFD.

Extension to negative numbers

With this basic definition of the arithmetic derivative in mind, we can begin to extend it a little bit. First, we can extend it to negative numbers in the following way. Given that we know the following:

$$(-x)^2 = x^2 \leftarrow x \in \mathbb{Z}_+$$

We can then use the Leibniz rule with this equality

$$(-x)^2 = x^2 \to [(-x)(-x)]' = (x * x)'$$

$$(x * x)' = x(x') + x(x') = 2x(x')$$

At this point you might notice it looks very similar to a power rule, it in fact is – we will prove it very soon.

$$[(-x)(-x)]' = -2x(-x)' = 2x(x')$$

$$\to (-x)' = -(x)'$$

Extension to rational numbers

The arithmetic derivative can be extended to ration numbers using a similar method as above. This is done in three steps. First, we must prove the power rule that was mentioned earlier.

Given x^n what would $(x^n)'$ be?

Using the product rule with x n times, it is simple to find the answer. $(x^n)' = (xxxxx \dots n\ times)' = nx'(xxxxx..n - 1\ times)$. The factor of n here is due to the fact that on application of the product rule multiple times, each the arithmetic derivative would be applied to each x one at a time (with the rest being left untouched hence the n-1) – this is similar to the beginning of the proof the arithmetic derivative acts like a function. X times itself n-1 times is the same as saying x^{n-1}. We can now say that $(x^n)' = x' * n * x^{n-1}$.

With that done, we can move onto step two. Negative exponents. We can show the power rule above also applies for negative exponents in the following way. But first, we must find the arithmetic derivative of 1. It is not prime and does not have any prime factors, so it is in a unique situation. To find the arithmetic derivative of one we can simply look at the following $(1 * 2)' = 1' * 2 + 2' * 1$ (Leibniz rule) $= 2' = 1 \to 1' * 2 + 1 = 1 \to 1' = 0$

The rest of this step is now simple. No words needed, just simple algebra

$$\frac{1'}{x} = (x^{-1})' \rightarrow \left(x * \frac{1}{x}\right)' = \frac{(x)'}{x} + x\left(\frac{1}{x}\right)' = (1)' = 0 \rightarrow x\left(\frac{1}{x}\right)' = -\frac{(x)'}{x} \rightarrow \left(\frac{1}{x}\right)' = -\frac{(x)'}{x^2}$$

$$= -1 * x' * x^{-2}$$

As this case satisfies the rule, multiplying by a positive power will result in negative powers. It is simple to show that the rule would still follow the rule. It will not be outlined here as it is a known result which does not build the intuition which helps with the novel ideas which are to be introduced.

Finally, the third step involves much the same argument, using the fact that the nth root of x, to the power of n is one. Like above this will not be outlined above.

Using simple algebraic methods like this, the Arithmetic derivative can be extended to all rational powers of rational numbers, but it will not be proved here. With the novel ideas that are to be introduced in the next section this fact will be shown in a very different and more concise way.

Novel ideas

The arithmetic derivative thus far as been thought of, quite accurately as a function. However, it can be defined in much more general terms. In order to see how this could be done, we can first extend the analogy of the arithmetic derivative to the normal derivative. Right now, we only have a product rule to compare the two as of now. The next rule that comes to mind when thinking of the derivative is the chain rule. I order to have a chain rule we must have a concept of a variable. In order to not confuse notation, we will use the symbol $\mathcal{A}$ to be analogous to the d in normal differentiation.

To be precise, from now on we will use the following notation:

$$\frac{\mathcal{A}y}{\mathcal{A}x}$$

To mean the arithmetic derivative of y with respect to x.

The first thing we must realise in order to assign a definition of a variable to the arithmetic derivative is that for some x or set of x's $\frac{\mathcal{A}y}{\mathcal{A}x}$ must produce the arithmetic derivative we are accustomed to. How would we find such a set of x's?

First recall the definition of the chain rule

$$\frac{dy}{du}\frac{du}{dx} = \frac{dy}{dx}$$

If we apply this to the notation we see above, we get the following

$$\frac{\mathcal{A}y}{\mathcal{A}u}\frac{\mathcal{A}u}{\mathcal{A}x} = \frac{\mathcal{A}y}{\mathcal{A}x}$$

To find a concept of 'variable' we will need to find out what variable we are differentiating with respect to in the traditional arithmetic derivative function. To do this, we notice that we can rearrange the formula in the following sense assuming x is the value which the traditional arithmetic derivative is with respect to, and changing the d to an $\mathcal{A}$ in order differentiate from normal differentiation:

$$\frac{\frac{\mathcal{A}y}{\mathcal{A}x}}{\frac{\mathcal{A}u}{\mathcal{A}x}} = \frac{\mathcal{A}y}{\mathcal{A}u} = \frac{y'}{u'}$$

With this it becomes immediately obvious that for $\frac{dy}{du} = y'$. That u' would be the same as the variable which defines the normal arithmetic derivative $(u' = x)$ and would have the requirement that $u' = 1$ which according to our definition above, means that u would be a prime. With this in mind, we will now define:

$$\mathcal{P} = An\ abitraty\ prime\ number$$

$$\frac{\mathcal{A}x}{\mathcal{AP}} = \frac{x'}{\mathcal{P}'} = \frac{x'}{1} = x'$$

From this point on, x' will be shorthand for $\frac{dx}{d\mathcal{P}}$.

Now we have tackled the concept of a more generalised arithmetic derivative, we can now use a similar notation for the arithmetic integral. If we define

$$y = x'$$

$$y = \frac{\mathcal{A}x}{\mathcal{AP}}$$

Then we can use the following notation to show the same equality

$$\int y\ \mathcal{AP} = x$$

Along with the definition of $\mathcal{P}$ being a prime number, we will denote $\mathcal{P}_x$ to be prime 'in' x. I.e. $\frac{\mathcal{AP}_x}{\mathcal{A}x} = 1$

We will finally define ε_x to denote the following set $\{N| \frac{\mathcal{A}N}{\mathcal{A}x} = N\}$ for any arbitrary x. In simpler terms, the set of numbers which are the arithmetic derivative of themselves. It has been proved previously (cite here) that $\varepsilon_{\mathcal{P}} = \{a^a | a \in \mathbb{P}\}$

Differential equations in $\mathcal{P}$

With our newfound notation we can now to begin to ask, what is the general solution to differential equations. We'll start with differential equations in $\mathcal{P}$ as it will help greatly with future endeavours. We will first use an identity

We will first start with simple differential equations of the form:

$$a\frac{\mathcal{A}x}{\mathcal{AP}} + bx = c$$

We can now divide through by a, define $p = \frac{b}{a}, q = \frac{c}{a}$ and use a method similar to the integration factor for normal differential equations. We will separate px into $(p-1)x + x$ and then we will multiply by a member of $\varepsilon_{\mathcal{P}}$.

$$\varepsilon_{\mathcal{P}}\frac{\mathcal{A}x}{\mathcal{AP}} + \varepsilon_{\mathcal{P}}x + (p-1)\varepsilon_{\mathcal{P}}x = \varepsilon_{\mathcal{P}}q$$

And using the multiplication rule in reverse we can deduce.

$$\frac{\mathcal{A}}{\mathcal{AP}}(\varepsilon_{\mathcal{P}}x) + (p-1)(\varepsilon_{\mathcal{P}}x) = \varepsilon_{\mathcal{P}}q$$

We can continue this process until we are left with the following differential equation. Note for this process to work, p must be a rational number.

$$\frac{\mathcal{A}}{\mathcal{AP}}(\varepsilon_{\mathcal{P}}{}^{p}x) = \varepsilon_{\mathcal{P}}{}^{p}q$$

We can then find a formula for x by taking the integral and dividing

$$x = \varepsilon_{\mathcal{P}}{}^{-p}\int \varepsilon_{\mathcal{P}}{}^{p}q\mathcal{AP} = \varepsilon_{\mathcal{P}}{}^{-\frac{b}{a}}\int \frac{c}{a}\varepsilon_{\mathcal{P}}{}^{\frac{b}{a}}\mathcal{AP}$$

Note: the easiest way to find the arithmetic integral of a number is by finding two prime numbers that add to it, and then multiply them giving the following identity. This is only possible for all integer's greater than 4 if the Goldbach Conjecture is true.

$$\int \mathcal{P}_1 + \mathcal{P}_2\mathcal{AP} = \mathcal{P}_1\mathcal{P}_2$$

Identity set of all x

With knowledge of the identity set ε_p containing all numbers that are their own arithmetic derivatives with respect to the primes. Further with the knowledge of how to solve a differential equation in $\mathcal{P}$. We can find a way to construct identity sets ε_x for all x that is differentiable in the primes.

The argument is as follows, each element y of our set ε_x must satisfy the following equality:

$$\frac{\mathcal{A}y}{\mathcal{A}x} = y$$

We can then use the chain rule to write this in terms of differentiation with respect to the primes

$$\frac{\mathcal{A}y}{\mathcal{A}x} = \frac{\mathcal{A}y}{\mathcal{AP}} * \frac{\mathcal{AP}}{\mathcal{A}x} = y$$

Following along this chain of thought we can use the identity above * to realise that $\frac{\mathcal{AP}}{\mathcal{A}x}$ is none other than $\frac{1}{\frac{\mathcal{A}x}{\mathcal{AP}}}$. Multiplying through by $\frac{\mathcal{A}x}{\mathcal{AP}}$ and subtracting $y\frac{\mathcal{A}x}{\mathcal{AP}}$ from both sides gives us a differential equation in y.

$$\frac{\mathcal{A}y}{\mathcal{AP}} - \frac{\mathcal{A}x}{\mathcal{AP}}y = 0$$

From here we can just use the formula from * and substitute $a = 1, b = -\frac{\mathcal{A}x}{\mathcal{AP}}$, and $c = 0$. Solving this differential equation (which with the c equal to zero term is trivial) gives the following solution y, of which each possible solution is a member of ε_x:

$$\varepsilon_x \ni \varepsilon_{\mathcal{P}}{}^{\frac{\mathcal{A}x}{\mathcal{AP}}}$$

The same argument can be made to convert the identity set from any one variable to another.

Differential equations in any y

It is quite simple to find a general solution to the following differential equation:

$$a\frac{\mathcal{A}x}{\mathcal{A}y} + bx = c$$

Using one of two methods. Firstly, one could use the chain rule to change the variable of y to p. Secondly, and less cumbersomely, one could use the same argument made before using an element of the set ε_y instead of $\varepsilon_{\mathcal{P}}$. Which will give you the following solution for x

$$x \;=\; {\varepsilon_{\mathcal{P}}}^{-p} \int {\varepsilon_{\mathcal{P}}}^{p} q \mathcal{A}y \;=\; {\varepsilon_y}^{-\frac{b}{a}} \int \frac{c}{a} {\varepsilon_y}^{\frac{b}{a}} \mathcal{A}y$$

Which can be rewritten (using the chain rule) in terms of $\mathcal{P}$.

$$x \;=\; {\varepsilon_{\mathcal{P}}}^{-p} \int {\varepsilon_{\mathcal{P}}}^{p} q \frac{\mathcal{A}y}{\mathcal{A}\mathcal{P}} \mathcal{A}\mathcal{P} \;=\; {\varepsilon_y}^{-\frac{b}{a}} \int \frac{c}{a} {\varepsilon_y}^{\frac{b}{a}} \frac{\mathcal{A}y}{\mathcal{A}\mathcal{P}} \mathcal{A}\mathcal{P}$$

This form can make use of the 'two prime rule' shown above *.

Reverse chain rule

$$\int k * \frac{\mathcal{A}x}{\mathcal{A}\mathcal{P}} \mathcal{A}\mathcal{P} = \int k * \mathcal{A}x$$

This part needs extending*

The Arithmetic Derivative and Complex Numbers

The arithmetic derivative can be extended to roots of unity in the following way. Call the Nth root of unity Z_N. i.e.

$Z_N^N - 1 = 0, \;\; N \in \mathbb{Z}_+$ One can use this property and the power rule to find the arithmetic derivative of Z_N.

$$\frac{\mathcal{A}}{\mathcal{A}\mathcal{P}} Z_N^N = N Z^{N-1} \frac{\mathcal{A}}{\mathcal{A}\mathcal{P}} Z_N = \frac{\mathcal{A}}{\mathcal{A}\mathcal{P}} 1 = 0$$

From this, it is easy to see that

$$\frac{\mathcal{A}}{\mathcal{A}\mathcal{P}} Z_N = 0$$

And at least to me, it is not immediately obvious what kind of implications this has. But after a few minutes of introspection, and the fact that all roots of unity lay on the unit circle. One can figure, that in general – at least most numbers that lay on the unit circle have the arithmetic derivative of zero. From this point on, we'll call these points zero points. We can then see that for these points of unity, there are infinite 'lines' going through the point, where the arithmetic derivative of all points on that line, is the same as the arithmetic derivative of the number going through the reals with the same magnitude, multiplied by the zero point the line goes through.

$$\frac{\mathcal{A}}{\mathcal{A}\mathcal{P}} K Z_N = \frac{\mathcal{A}K}{\mathcal{A}\mathcal{P}} * Z_N$$

Using the multiplication rule and using the fact the arithmetic derivative of the zero point is zero.

This can easily be extended to the arithmetic derivative with respect to an arbitrary X. The arithmetic derivative of zero points, are zero (put a proof here using x*1=x) therefore:

$$\frac{\mathcal{A}}{\mathcal{A}x} K Z_N = \frac{\mathcal{A}K}{\mathcal{A}x} * Z_N$$

A second way of looking at multiple variables

So far, we have discussed the concept of multiple variables existing within the context of the arithmetic derivative using the definition of the prime derivative as a baseline. Which forbids defining something like the arithmetic derivative with respect to a gaussian prime. Luckily, this is not the most general definition we can have. There is another, far superior definition. However, this is far removed from what we have looked at so far. And very abstract. But, consequentially allows for quicker, more general proofs and some mind-bending results.

To find this more general definition, as always, we look at our current definition at try to change anything that is arbitrary.

Looking at the definition the first thing we find to be arbitrary is our defining of the arithmetic derivative to be one at the prime numbers. This is arbitrary in two ways, the fact it is defined as one, and the fact it is defined at this value. Let us use some new, unusual notation to define 'an' arithmetic derivative, which maps a set of numbers to another set of numbers using the chain and multiplication rule above, but with arbitrary starting points.

Let

$$S(=T)$$

Denote the arithmetic derivative (again an arithmetic derivative being, at its core – something that follows the chain and multiplication rule), which for every element of the set S maps to every element in set T.

To see why this notation and definition is used, imagine you define an arithmetic derivative such that $\frac{\mathcal{A}2}{\mathcal{A}2}=1$ and $\frac{\mathcal{A}3}{\mathcal{A}3}=1$. Application of the chain rule here will see that all combinations of $\mathcal{A}2$ and $\mathcal{A}3$ will also result in the arithmetic derivative at that point to be 1. This is very important and is the basis for which this entire definition lays upon.

$$\frac{\mathcal{A}y}{\mathcal{A}x}=\frac{\mathcal{A}w}{\mathcal{A}z}=T, \leftarrow w,y,x,z \in S$$

This will always hold.

Let us for convenience also define an equivalence relation between Arithmetic derivatives. In order to do that we must define what a subset of the Arithmetic derivative is.

$$S(=T) \subset U(=V)$$

If and only if

$$(S \subset U) \wedge (T \subset V)$$

With this we can now define our equivalence relation. Arithmetic derivatives A and B are said to be equivalent if, and only if:

$$(A \subset B) \wedge (B \subset A) \rightarrow A \sim B$$

Or more primitively, the same thing, if the arithmetic derivatives $A = S(=T)$ and $B = U(=V)$ are said to be equal if:

$$(S \subset U) \wedge (T \subset V) \wedge (U \subset S) \wedge (V \subset T) \rightarrow A \sim B$$

Such rules will allow for the creation of two different types of Arithmetic derivative, which we will call N and C. N type arithmetic derivatives create a set of single valued functions, while C type arithmetic derivatives create a set of multivalued functions.

It should be notice that the above notation, and the sets they contain are not the arithmetic derivatives. We are not looking at the sets – the sets are merely a way to describe an arithmetic derivative (Of which there are infinitely many).

Furthermore we can the binary relation $S(=s) * T(=t)$ to follow the following rule

$$\forall a,b \in S\ \ \forall c,d \in T:\ \frac{\mathcal{A}a}{\mathcal{A}b}=s \wedge \frac{\mathcal{A}c}{\mathcal{A}a}=s \wedge \frac{\mathcal{A}a}{\mathcal{A}c}=t \wedge \frac{\mathcal{A}c}{\mathcal{A}d}=t$$

If $S \cup T = \emptyset$ The function produced is an N(Single valued) type, otherwise it can produce a C(multivalued) type or an N type.

Using this definition, we can create all sorts of interesting arithmetic derivatives without running into problems. The arithmetic derivative we have been looking at thus far is constructed from $\{p \mid p \in \mathbb{P}\}(= 1)$. At the beginning of this document, we did not show the entire domain of this arithmetic derivative, that is because with our new definition it takes just minutes, and a few lines – rather than hours and a few pages. The argument is as follows

Given the arithmetic derivative defined by

$$A = \{x\}(= 1)$$

What is the range of A? Well using the chain and multiplication rule, we can only define it for rational powers of x. Or, more accurately – solutions to the polynomial $y^{\frac{p}{q}} = x$ where p and q are both integers, and q is nonzero. Why is this the case?

Suppose you apply the multiplication rule on some non-rational power of x, say x is two

$$A = \{2\}(= 1)$$

We apply our multiplication rule to find $\frac{\mathcal{A}3}{\mathcal{A}2}$. We must find a number which multiplied by two, gives three – and which is able to be defined using powers of two(as, in order to construct a number with the single value – we must only use that value, and numbers which can be created from multiplying that value – for example with two $\frac{\mathcal{A}4}{\mathcal{A}2} = 2\frac{\mathcal{A}2}{\mathcal{A}2} + 2\frac{\mathcal{A}2}{\mathcal{A}2} = 4$. The number must be able to be decomposed into a value in $S \mid A = S(= T)$ in finitely many 'moves. Which cannot be done irrational powers of a member of S.

Let's suppose we then use our binary relation on two arithmetic derivatives

$$A = \{2\}(= 1) * \{3\}(= 1)$$

For what numbers is this defined? Well, if we can define any rational power of two – and any rational power of three. We can use the multiplication rule to define those two powers multiplied. So it is easy to see that the domain of A would be (solutions to) $2^p 3^q$ where p and q are both rational numbers.

This argument can be repeated for every prime number, and the domain of the arithmetic derivative from $\{p \mid p \in \mathbb{P}\}(= 1)$ is every solution to

$$\prod_{p \in \mathbb{P}} p^q \leftarrow q \in \mathbb{Q}$$

With our binary operator defined, we can begin to see if we can find arithmetic derivatives with certain properties, do they form a group? A monoid? Since we have a binary operator – we can say for sure they form a monoid. In order for a semi-group or group however, we must have an identity element.

What would the requirement for an identity element be? It would have to, when added to A, produce A. Using our handy equivalence relation, we can easily see that our element must be a subset of all possible sets. There is only one set which this applies to, the empty set. So we can say, our identity element is:

$$i = \{\}(= \{\}) = \emptyset(= \emptyset)$$

As for a more intuitive reason for why there cannot be a non-empty set which is the identity element – you must consider the multiplication rule, if an nonempty arithmetic derivative, all of it's elements must produce an element of every possible arithmetic derivative when put through the multiplication rule, ignoring the fact that not every possible arithmetic derivative shares a value at a single point, the identity would have to contain a number which is both the multiplicative and additive identity – which doesn't exist.

There also exist identity elements for classes of arithmetic derivatives. For example

A quick note on multiplying arithmetic derivatives. We'll define three things – If an arithmetic derivative can be expressed via the binary operation (from this point on, product) of two other arithmetic derivatives – it will be a composite arithmetic derivative. If it cannot – it will be a prime arithmetic derivative. And if the set of infinitely many unique arithmetic derivatives (unique being – no arithmetic derivative description set left hand side contains a rational power of another). It will be a prime describing arithmetic derivative. As it will describe an infinite set of numbers which can said to be prime in some way, the Gaussian primes and the normal prime numbers are both prime describing arithmetic derivatives. Therefore, the prime describing arithmetic derivatives are infinitely composite.

All in all, we can abstractly define the arithmetic derivative in the following way:

$$A\ pair\ \ a = (S, n)\ where\ S \subset \mathfrak{P}(\mathbb{Z})\ and\ n\ \subset\ \mathbb{Z} \rightarrow S(= n)$$

With this we will also extend the two differing notations into one.

$$\frac{\mathcal{A}_T y}{\mathcal{A}_S x}$$

Will from this point on, mean the arithmetic derivative of y with respect to x, using the arithmetic derivative defined by S(=T).

What we have looked at so far can be described as analogous to 'partial derivatives' of a bigger whole. If we have two mutually disjoint arithmetic derivatives and take the arithmetic derivative of an element of one with respect to an element of another – you will find that you cannot define it. Given A = S(=T) and B = U(=V).

$$\frac{\mathcal{A}_S y}{\mathcal{A}_T x} \nrightarrow \frac{\mathcal{A}_U y}{\mathcal{A}_T x} \land \frac{\mathcal{A}_S y}{\mathcal{A}_V x} \nrightarrow \frac{\mathcal{A}_U y}{\mathcal{A}_V x}$$

However, a new concept can be introduced. If the arithmetic derivatives discussed thus far can be thought of as partial derivatives, you can create an analogy of the total derivative (Of S(=T) * U(=V)):

$$\frac{\mathcal{A}_{T*V} y}{\mathcal{A}_{S*U} \emptyset} = \sum_{x\, \in\, S} \frac{\mathcal{A}_T y}{\mathcal{A}_S x} + \sum_{x\, \in\, U} \frac{\mathcal{A}_U y}{\mathcal{A}_V x}$$

Arithmetic derivatives can be described that follow many different sets of rules. For example

$$\{x \mid x\ is\ odd\}(= 0) * \{2\}(= 1)$$

Defines the arithmetic derivative where for all even x, $x' = \frac{x}{2}$. Which is half of the Collatz function.

Unfortunately, the other half cannot, in my experience, be constructed. However, I am sure many cool functions can be described in terms of arithmetic differential equations.

---Unfinished---

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Notes: There is some unknown relation, involving the chain rule with normal functions.

If the derivative of a set of numbers with respect to x is one (x would be included in this set) then the derivative of a member of the set, with respect to any member of the set is also one * proof on paper. {1}(=1) describes infinity (no factors). If S contains x and a rational power of x, it is type C